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Curvature collineations of non-expanding and twist-free vacuum type-N metrics in general relativity

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Abstract. The curvature collineation equations have been solved for the two families of Petrov type-N plane-fronted gravitational wave solutions of Einstein's vacuum field equations in general relativity. Both of these solutions always have non-trivial curvature collineations, i.e. vector fields $\boldsymbol{\xi}$ with respect to which the components $R^{\mu}{}_{\nu\alpha\beta}$ of the Riemann tensor for those solutions are Lie derivable.

1. Introduction

Katzin *et al* (1969) defined a curvature collineation (CC) of a metric $\boldsymbol{g}$ to be a vector field $\boldsymbol{\xi}$ such that

$$\mathscr{L}_{\xi}R^{\mu}{}_{\nu\alpha\beta}=0 \tag{1.1}$$

where $R^{\mu}{}_{\nu\alpha\beta}$ are the components of the Riemann tensor of that metric $\boldsymbol{g}$, i.e. the Riemann tensor is Lie derivable along the congruence of curves with tangent vector $\boldsymbol{\xi}$. All special conformal motions (SCM) with $\boldsymbol{\xi}$ such that

$$\mathscr{L}_{\xi}g_{\mu\nu}=\phi g_{\mu\nu} \qquad \phi_{;\mu\nu}=0 \tag{1.2}$$

have been shown by Katzin, Levine and Davis to be CC's. However the converse is not true in general; there are some metrics with CC's which are not SCM's. A CC is said here to be non-trivial if it does not reduce to a SCM.

Collinson (1970) has shown that in vacuum, the only solutions of Einstein's field equations that have non-trivial CC's have Weyl tensor of Petrov type N. Katzin, Levine and Davis showed that Einstein space–times (with $R_{\mu\nu}=\frac{1}{4}Rg_{\mu\nu}\neq 0$) do not admit non-trivial CC's. Also Tariq and Tupper (1977) showed that CC's admitted by the source-free Einstein–Maxwell space–times are SCM's, except possibly in the case where the Maxwell field is null and the Weyl tensor is type N or O. Reasons are given elsewhere by McIntosh (1979), and in future papers by McIntosh and Halford, why extremely few space–times, even for other forms of the matter tensor in Einstein's equations, admit non-trivial CC's, and the possible cases will be listed in these papers. In this paper two vacuum space–times are discussed which admit non-trivial CC's.

A necessary condition for $\boldsymbol{\xi}$ to be a CC is (Katzin, Levine and Davis' equation (2.11)) that it satisfies the equations

$$x_{\mu\nu}R^{\nu}{}_{\lambda\alpha\beta}+x_{\lambda\nu}R^{\nu}{}_{\mu\alpha\beta}=0 \tag{1.3}$$

where

$$x_{\mu\nu} = \xi_{(u;\nu)}. \tag{1.4}$$

Equation (1.3) can easily be obtained by Lie differentiating the identity

$$g_{\mu\nu}R^{\nu}{}_{\lambda\alpha\beta} + g_{\lambda\nu}R^{\nu}{}_{\mu\alpha\beta} = 0 \tag{1.5}$$

with respect to $\boldsymbol{\xi}$ and using (1.1). Collinson (1970) has shown that in vacuum the only possible solutions of (1.3) for $x_{\mu\nu}$ are

$$x_{\mu\nu} = \phi g_{\mu\nu} + \alpha l_\mu l_\nu \tag{1.6}$$

where $\alpha = 0$ except for Petrov type-N solutions, in which case α may be non-zero, and $\boldsymbol{l}$ is the four-fold repeated principal null congruence of the Weyl tensor. Since in this case $\boldsymbol{l}$ satisfies

$$l_\mu C^{\mu}{}_{\nu\alpha\beta} = l_\mu R^{\mu}{}_{\nu\alpha\beta} = 0, \tag{1.7}$$

(1.6) obviously satisfies (1.3). Collinson gave one particular pp-wave vacuum solution for which $\alpha \neq 0$ and $\boldsymbol{\xi}$ was a non-trivial CC. Aichelburg (1972) gave a list, which unfortunately is not complete, of CC's given by the plane-fronted gravitational wave vacuum metric with line element

$$\mathrm{d}s^2 = [F(u, \zeta) + \bar{F}(u, \bar{\zeta})]\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}v - 2\,\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta} \tag{1.8}$$

where F is an arbitrary function of u and ζ. Where $(x^0, x^1, x^2, x^3) = (u, v, \zeta, \bar{\zeta})$, the only non-zero independent components of the Riemann tensor for the metric as given by (1.8) are

$$R^{1}{}_{202} = R^{3}{}_{002} = \tfrac{1}{2}F_{,\zeta\zeta} \tag{1.9}$$

and their complex conjugates $R^{1}{}_{303}$ and $R^{2}{}_{003}$.

Kundt (1961) gave two separate families of plane-fronted gravitational wave solutions of the vacuum Einstein field equations. One family is described by (1.8) and represents waves with parallel rays: the so-called pp metrics. In the other family the rays are not parallel and have non-zero rotation $|\Omega|$. Both have non-expanding principal null congruences. In Newman–Penrose (1962) language, $\rho = 0$ for both cases and $\tau = 0$ for (1.8) but $\tau(\equiv \Omega) \neq 0$ in the second case. The line element in this last case can be written as

$$\mathrm{d}s^2 = -\{v^2x^{-2} + (\zeta+\bar{\zeta})[F(u,\zeta) + \bar{F}(u,\bar{\zeta})]\}\,\mathrm{d}u^2 + 2\,\mathrm{d}u[\mathrm{d}v - vx^{-1}(\mathrm{d}\zeta + \mathrm{d}\bar{\zeta})] - \mathrm{d}\zeta\,\mathrm{d}\bar{\zeta} \tag{1.10}$$

where

$$2x = \zeta + \bar{\zeta} \tag{1.11}$$

and F is an arbitrary complex function of u and ζ. The only non-zero independent components of the Riemann tensor for (1.10) are

$$R^{1}{}_{002} = 2vF_{,\zeta\zeta}, \qquad R^{1}{}_{202} = xF_{,\zeta\zeta}, \qquad R^{3}{}_{002} = 2xF_{,\zeta\zeta} \tag{1.12}$$

and their complex conjugates $R^{1}{}_{003}$, $R^{1}{}_{303}$ and $R^{2}{}_{003}$.

Curvature collineations for both these metrics are listed in this paper and an interpretation is given for those of the $\tau \neq 0$ metric (1.10).

2. The $\tau \neq 0$ metric (1.10,11)

The necessary conditions (1.6) that the metric given by (1.10,11) admit a curvature collineation (1.1) are that

$$\mathscr{L}_{\boldsymbol{\xi}} g_{\mu\nu} = \phi g_{\mu\nu} + \alpha \delta^0{}_\mu \delta^0{}_\nu \tag{2.1}$$

where $\delta^0{}_\mu = l_\mu$ are the components of the repeated principal null congruence of the metric in those coordinates. It is easiest first to solve (2.1) for the necessary conditions on $\boldsymbol{\xi}$ and then to solve the rest of the full set of equations (1.1). Equations (2.1) give

$$\begin{gathered}\boldsymbol{\xi} = B(u)\partial_u + \{[2a - B'(u)]v + E(u)(\zeta+\bar{\zeta})^2\}\partial_v + (a\zeta + \mathrm{i}b)\partial_\zeta + (a\bar{\zeta} - \mathrm{i}b)\partial_{\bar{\zeta}} \\ \phi = 2a \\ \alpha(u, v, \zeta, \bar{\zeta}) = 2E'(u)(\zeta+\bar{\zeta})^2 + (\zeta+\bar{\zeta})G(u, \zeta, \bar{\zeta}) - 2v[4E(u) + B''(u)]\end{gathered} \tag{2.2}$$

where

$$G(u, \zeta, \bar{\zeta}) = [a - 2B'(u)](F + \bar{F}) - (a\zeta + \mathrm{i}b)F_{,\zeta} - (a\bar{\zeta} - \mathrm{i}b)\bar{F}_{,\bar{\zeta}} - B(u)(F_{,u} + \bar{F}_{,u}). \tag{2.3}$$

Here a and b are arbitrary real constants, B and E are arbitrary real functions of u and F stands for $F(u, \zeta)$.

However, $\boldsymbol{\xi}$ as given in (2.2) does not define a CC, as only the necessary conditions (2.1) have been satisfied. When this $\boldsymbol{\xi}$ is substituted into (1.1) it is found that F has to satisfy

$$(a\zeta + \mathrm{i}b)F_{,\zeta\zeta\zeta} + [a + 2B'(u)]F_{,\zeta\zeta} + B(u)F_{,u\zeta\zeta} = 0.$$

This can be integrated to yield

$$[a - 2B'(u)]F - (a\zeta + \mathrm{i}b)F_{,\zeta} - B(u)F_{,u} = \gamma(u)\zeta + \tfrac{1}{2}H(u) + \mathrm{i}J(u) \tag{2.4}$$

where γ is an arbitrary complex function of u and H and J are arbitrary real functions of u. Then $G(u, \zeta, \bar{\zeta})$ can be replaced by

$$G(u, \zeta, \bar{\zeta}) = \gamma(u)\zeta + \bar{\gamma}(u)\bar{\zeta} + H(u) \tag{2.5}$$

so that α in (2.1) and (2.2) is given by

$$\alpha(u, v, \zeta, \bar{\zeta}) = (\zeta+\bar{\zeta})[\gamma(u)\zeta + \bar{\gamma}(u)\bar{\zeta} + H(u)] + 2E'(u)(\zeta+\zeta)^2 - 2v[4E(u) + B''(u)]. \tag{2.6}$$

CC's are now given by (2.2) and (2.5). In the general case when F is arbitrary, (2.4) gives $a = b = 0$, $B = H = J = \gamma = 0$; however there is always the CC given by

$$\boldsymbol{\xi} = E(u)(\zeta+\bar{\zeta})^2\partial_v, \qquad \alpha = -8vE(u) + 2E'(u)(\zeta+\bar{\zeta})^2. \tag{2.7}$$

Equation (2.4) can be solved to find explicit forms of $F(u, \zeta)$. However, a number of canonical cases arise and it is not worthwhile to list them. Also in practice the procedure works the other way round: if an explicit form of $F(u, \zeta)$ is given, equation (2.4) soon yields a, b, B, H, J and γ and the precise form of the CC is known.

Special conformal motions of the metric thus have

$$\alpha = 0, \qquad E = 0, \qquad \gamma = 0, \qquad H = 0, \qquad B''(u) = 0, \tag{2.8}$$

and F satisfies the differential equation

$$G(u, \zeta, \bar{\zeta}) = 0. \tag{2.9}$$

Since ϕ is a constant, these motions are either non-trivial homothetic ones ($a \neq 0$) or Killing ones ($a = 0$).

3. Description of curvature collineations for the $\tau \neq 0$ metric

The non-trivial CC's of the $\tau \neq 0$ metric are described by $\boldsymbol{\xi}$ as given by (2.2). The Lie derivative of $g_{\mu\nu}$ with respect to $\boldsymbol{\xi}$ is, from (2.1), (2.2) and (2.5),

$$\mathscr{L}_{\xi} g_{00} = 2ag_{00} + (\zeta + \bar{\zeta})[\gamma(u)\zeta + \bar{\gamma}(u)\bar{\zeta} + H(u)] + 2E'(u)(\zeta + \bar{\zeta})^2 - 2v[4E(u) + B''(u)] \tag{3.1}$$

and

$$\mathscr{L}_{\xi} g_{\mu i} = 2ag_{\mu i} \qquad \text{for } i = 1, 2, 3, \tag{3.2}$$

where F satisfies the differential equation (2.4). The 'a' term just describes a constant scaling of the metric coefficients; with all the constants and arbitrary functions in $\boldsymbol{\xi}$ zero except for a, $\boldsymbol{\xi}$ is just a homothetic vector field. There is thus no real loss of interpretation of the CC's in just considering the case with $a = 0$.

The infinitesimal effect on g_{00} under the Lie derivative $\boldsymbol{\xi}$ is given by (3.1). The corresponding finite transformations affect g_{00} in a similar way, namely

$$g_{00} \rightarrow g_{00} + vP(u) + (\zeta + \bar{\zeta})L(u, \zeta) \tag{3.3}$$

where

$$L(u, \zeta) = 2Q(u) + \zeta\delta(u) + \bar{\zeta}\bar{\delta}(u). \tag{3.4}$$

Here P and Q are real functions of u and δ is a complex function of u. The relationships between these functions and those in $\boldsymbol{\xi}$ can readily be obtained. Now the form of the original metric (1.10) has

$$g_{00} = -v^2 x^{-2} - (\zeta + \bar{\zeta})[F(u, \zeta) + \bar{F}(u, \bar{\zeta})]. \tag{3.5}$$

If, instead, g_{00} was given by

$$g_{00} = -v^2 x^{-2} + 8vM(u) - (\zeta + \bar{\zeta})[F_1(u, \zeta) + \bar{F}_1(u, \bar{\zeta})], \tag{3.6}$$

where M is an arbitrary function of u, then M can be eliminated from g_{00} in (3.6) by a coordinate transformation of the form

$$\tilde{v} = v + M(u)(\zeta + \bar{\zeta})^2, \qquad \tilde{u} = u, \qquad \tilde{\zeta} = \zeta, \tag{3.7}$$

and where

$$F_1(u, \zeta) = F(u, \zeta) + 2\zeta[M'(u) + 2M^2(u)]. \tag{3.8}$$

If the more general form (3.6) is taken rather than (3.5), then M and its derivatives do not appear in the components of the Riemann tensor. As the non-zero $R^{\mu}{}_{\nu\alpha\beta}$ are all proportional to $F_{,\zeta\zeta}$ or its complex conjugates, these components are also unaffected by a change in $F(u, \zeta)$ of the form (3.8) (since $F_{,\zeta\zeta} = F_{1,\zeta\zeta}$).

Thus any (non-trivial) CC consists of a scaling or homothetic motion plus a mapping which changes g_{00} as in (3.3), (3.4). It is obvious from the above discussion that the components $R^{\mu}{}_{\nu\alpha\beta}$ of the Riemann tensor are unaffected by these changes. It is interesting that there are no other CC's than the ones giving these obvious changes.

4. The $\tau = 0$ metric (1.8)

For the pp-wave metric the equations (2.1), the necessary conditions for a CC which satisfy (1.3), can be solved but the solutions are rather messy. It is best not to write them down explicitly but to solve them and the other equations for a CC (1.1) together, i.e. in essence to solve all of the equations (1.1) together. This yields

$$\begin{aligned}\boldsymbol{\xi} &= P(u)\partial_u + \{v[2N(u)-P'(u)] + N'(u)\zeta\bar{\zeta} + \bar{\beta}'(u)\zeta + \beta'(u)\bar{\zeta} + M(u)\}\partial_v \\ &\quad + \{[N(u)+\mathrm{i}b]\zeta + \beta(u)\}\partial_\zeta + \{[N(u)-\mathrm{i}b]\bar{\zeta} + \bar{\beta}(u)\}\partial_{\bar{\zeta}} \\ &\quad \phi(u) = 2N(u) \\ \alpha &= 2v[2N'(u) - P''(u)] + 2N''(u)\zeta\bar{\zeta} + 2M'(u) + K(u) \\ &\quad + [\gamma(u) + 2\bar{\beta}''(u)]\zeta + [\bar{\gamma}(u) + 2\beta''(u)]\bar{\zeta}\end{aligned} \tag{4.1}$$

and, where F stands for $F(u, \zeta)$, the differential equation for F

$$2[P'(u)-N(u)]F + P(u)F_{,u} + \{[N(u)+\mathrm{i}b]\zeta + \beta(u)\}F_{,\zeta} = \gamma(u)\zeta + \tfrac{1}{2}K(u) + \mathrm{i}J(u). \tag{4.2}$$

Here b is an arbitrary real constant, K, J, M, N and P are arbitrary real functions of u and β and γ are arbitrary complex functions of u.

$\boldsymbol{\xi}$ in (4.1) becomes a SCM when

$$\alpha = 0 \tag{4.3}$$

in which case (4.1) reduces to

$$\begin{aligned}\boldsymbol{\xi} &= (cu^2 + eu + a)\,\partial_u + [v(2d-e) + c\zeta\bar{\zeta} + \bar{\beta}'(u)\zeta + \beta'(u)\bar{\zeta} + M(u)]\,\partial_v \\ &\quad + [(cu+d+\mathrm{i}b)\zeta + \beta(u)]\,\partial_\zeta + [(cu+d-\mathrm{i}b)\bar{\zeta} + \bar{\beta}(u)]\,\partial_{\bar{\zeta}} \\ &\quad \phi = 2(cu+d)\end{aligned} \tag{4.4}$$

and F satisfies

$$2(cu+e-a)F + (cu^2+eu+a)F_{,u} + [(cu+d+\mathrm{i}b)\zeta + \beta(u)]F_{,\zeta} = -2\bar{\beta}''\zeta - M'(u) + \mathrm{i}J(u). \tag{4.5}$$

Here a, b, c, d and e are arbitrary real constants.

Again these differential equations for F can be solved and a list of canonical classes for F given. A list where $\boldsymbol{\xi}$ is an isometry and $\phi = 0$ in (4.4) is given by Ehlers and Kundt (1962; table 2—5.1). Collinson (1970) gave one case where F is such that the metric admits a non-trivial CC. Katzin *et al* (1970) showed that the general pp metric always admits a non-trivial CC of the form (4.6) below. Aichelburg (1972) also establishes this fact and gives some forms of F for which the metric admits more general non-trivial CC's. Unfortunately his list is incomplete; for example, one case he does not mention is $F(u, \zeta) = \exp[\lambda(u)\zeta]$, where λ is an arbitrary function of u. Obviously a non-trivial $\boldsymbol{\xi}$ can be found from (4.1) and (4.2) for this F.

As in the case of the $\tau \neq 0$ metric, in practice the substitution of a given particular form of F into (4.2) or (4.5) yields the possible non-zero constants and functions of u that enable $\boldsymbol{\xi}$ to be written down explicitly.

As mentioned above, the general metric always admits a non-trivial CC since, with F arbitrary, all the unknowns in the expression for $\boldsymbol{\xi}$ in (4.1) are zero except for $M(u)$. In this case

$$\boldsymbol{\xi} = M(u)\partial_v, \qquad \alpha = 2M'(u). \tag{4.6}$$

This is equivalent to $v \to v + M(u)$, $u \to u$, $\zeta \to \zeta$. It is obvious that the complex components (1.9) of the Riemann tensor (and their complex conjugates) are unaltered under this mapping. When Katzin *et al* (1970) showed that the metric admitted a CC of this type, they also showed that, for special forms of the function M, the CC's are trivial ones. However, M can be arbitrary and the CC is then non-trivial.

5. Conclusion

The curvature collineation equations (1.1) have been solved for both the type-N plane-fronted gravitational wave metric (1.8) and (1.10, 11). In both cases the metrics always admit CC's which are not SCM's (and incidentally are not also affine collineations) and quite often other non-trivial CC's as well. The SCM's admitted by both of these metrics are also given as obvious subcases of the CC's.

The equations (1.3), which here arise as necessary conditions for the vector ξ satisfying (1.4) to be a CC, will be discussed later by McIntosh and Halford. This is in relationship to the holonomy group and the problem of finding $x_{\mu\nu}$ for a given set of $R^{\mu}{}_{\nu\alpha\beta}$ which are the components of the Riemann tensor for some metric, and extends the work of Hlavatý (1959a,b) and Ihrig (1975)—see also McIntosh (1979). Some of the results in this paper will be used as illustrations of results in these further papers.

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References

Aichelburg P C 1972 *Gen. Rel. Grav.* **3** 397
Collinson C D 1970 *J. Math. Phys.* **11** 818
Ehlers J and Kundt W 1962 *Gravitation: An Introduction to Current Research* ed L Witten (New York: Wiley) ch 2
Hlavatý 1959a *J. Math. Mech.* **8** 285
—— 1959b *J. Math. Mech.* **8** 597
Ihrig E 1975 *J. Math. Phys.* **16** 54
Katzin G H, Levine J and Davis W R 1969 *J. Math. Phys.* **10** 617
—— 1970 *J. Math. Phys.* **11** 1578
Kundt W 1961 *Z. Phys.* **163** 77
McIntosh C B G 1979 *Symmetries and Exact Solutions of Einstein's Equations, Preprint*
Newman E T and Penrose R 1962 *J. Math. Phys.* **3** 566
Tariq N and Tupper B O J 1977 *Tensor* **31** 42